Partial Solution Set, Leon §4.1

4.1.1 For each of five transformations, we are to verify linearity and describe geometrically the effect of the transformation. Verification is straightforward for all; a sketch might make it easier to see what's going on geometrically.

(a) $L(\mathbf{x}) = (-x_1, x_2)^T$. Verifying linearity is simple: we simply show that

$$\begin{aligned} L(\alpha\mathbf{x}+\mathbf{y}) &= (-\alpha x_1 + y_1, \alpha x_2 + y_2)^T \\ &= (-\alpha x_1, \alpha x_2)^T + (y_1, y_2)^T \\ &= \alpha(x_1, x_2)^T + (y_1, y_2)^T \\ &= \alpha L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

The geometric effect is reflection across the x_2-axis.

(c) Reflection across the identity line $x_1 = x_2$.

(e) Projection onto the x_2-axis.

4.1.2 Let L be the linear transformation mapping $\mathbf{R}^2$ into itself defined by

$$L(\mathbf{x}) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha)^T.$$

Express x_1, x_2, and $L(\mathbf{x})$ in terms of polar coordinates. Describe geometrically the effect of the linear transformation.

Solution: Rewriting in terms of polar coordinates, we have $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$. But now we have

$$\begin{aligned} L(\mathbf{x}) &= (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha)^T \\ &= (r\cos\theta\cos\alpha - r\sin\theta\sin\alpha, r\cos\theta\sin\alpha + r\sin\theta\cos\alpha)^T \\ &= (r\cos(\theta+\alpha), r\sin(\theta+\alpha))^T, \end{aligned}$$

from which we see that the original vector has been rotated counterclockwise through an angle α.

4.1.3 Let $\mathbf{a}$ be a fixed nonzero vector in $\mathbf{R}^2$. A mapping of the form $L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ is called a *translation*. Show that a translation is not a linear transformation. Illustrate geometrically the effect of a translation.

Solution: The verification that L is nonlinear is easy:

$$\begin{aligned} L(\alpha\mathbf{x}+\mathbf{y}) &= \alpha\mathbf{x} + \mathbf{y} + \mathbf{a} \\ &\neq \alpha(\mathbf{x}+\mathbf{a}) + (\mathbf{y}+\mathbf{a}) \\ &= \alpha L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

The geometric effect is to shift the line containing $\mathbf{x}$ away from the origin, the distance and direction of the shift determined by the length and direction of $\mathbf{a}$. In the event that $\mathbf{a}$ and $\mathbf{x}$ are collinear, the shifted line still passes through the origin, but the transformation is nonetheless nonlinear, as shown by the preceding step.

4.1.6 Determine whether the following are linear transformations from $\mathbf{R}^2$ into $\mathbf{R}^3$.

(a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$. This is nonlinear, since

$$L(\alpha\mathbf{x}) = (\alpha x_1, \alpha x_2, 1)^T \neq (\alpha x_1, \alpha x_2, \alpha)^T = \alpha L(\mathbf{x}).$$

(c) $L(\mathbf{x}) = (x_1, 0, 0)^T$. This is linear, since

$$L(\alpha\mathbf{x} + \mathbf{y}) = (\alpha x_1 + y_1, 0, 0)^T = \alpha(x_1, 0, 0)^T + (y_1, 0, 0)^T = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

4.1.14 Let L be a linear operator mapping a vector space V into itself. Recursively define L^n, by $L^1 = L$ and $L^{n+1}(\mathbf{v}) = L(L^n(\mathbf{v}))$ for all $n \geq 1$ and all $\mathbf{v} \in V$. Show that L^n is a linear operator on V for each $n \geq 1$.

Proof: The proof is by induction on n. By definition of L^n, the result holds for $n = 1$. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}$, let α be a scalar, and let $k \geq 1$. Then

$$\begin{aligned}
L^{k+1}(\alpha\mathbf{u} + \mathbf{v}) &= L(L^k((\alpha\mathbf{u} + \mathbf{v}) \\
&= L(\alpha L^k(\mathbf{u}) + L^k(\mathbf{v}) \\
&= \alpha L(L^k(\mathbf{u})) + L(L^k(\mathbf{v}) \\
&= \alpha L^{k+1}(\mathbf{v}) + L^{k+1}(\mathbf{v}), \text{ and we're done.} \qquad \square
\end{aligned}$$

4.1.15 Let $L_1 : U \to V$ and $L_2 : V \to W$ be linear transformations and let $L = L_2 \circ L_1$ be the mapping defined by $L(\mathbf{u}) = L_2(L_1(\mathbf{u}))$ for each $\mathbf{u} \in U$. Show that L is a linear transformation mapping U into W.

Solution: That L is a mapping from U into W follows from elementary properties of functions. We must show that L is linear. So let $\mathbf{u}_1, \mathbf{u}_2 \in U$, and let α be a scalar. Then

$$\begin{aligned}
L(\alpha\mathbf{u}_1 + \mathbf{u}_2) &= L_2\left(L_1(\alpha\mathbf{u}_1 + \mathbf{u}_2)\right) && \text{(Definition of } L\text{)} \\
&= L_2\left(\alpha L_1(\mathbf{u}_1) + L_1(\mathbf{u}_2)\right) && \text{(linearity of } L_1\text{)} \\
&= \alpha L_2\left(L_1(\mathbf{u}_1)\right) + L_2\left(L_1(\mathbf{u}_2)\right) && \text{(linearity of } L_2\text{)} \\
&= \alpha L(\mathbf{u}_1) + L(\mathbf{u}_2) && \text{(Definition of } L.\text{)} \qquad \square
\end{aligned}$$

4.1.16 Determine the kernel and range of each of the following linear transformations from $\mathbf{R}^3$ into itself.

(a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$. The kernel is the zero vector from $\mathbf{R}^3$, since $L(\mathbf{x}) = \mathbf{0}$ if and only if $x_1 = x_2 = x_3 = 0$. The range is all of $\mathbf{R}^3$: let $\mathbf{y} = (y_1, y_2, y_3)^T \in \mathbf{R}^3$. Then $\mathbf{y} = L\left((y_3, y_2, y_1)^T\right)$, so L is an onto mapping.

(c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$. The kernel contains all vectors of the form $(0, x_2, x_3)$, and is therefore a two-dimensional subspace of $\mathbf{R}^3$. The range is $\text{Span}\left((1,1,1)^T\right)$.